Dynamics of Electromechanical Systems

Conference Paper · March 1996

DOI: 10.13140/2.1.2607.8086

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Abstract

Electromechanical systems can be regarded as physical structures characterized by interaction of electromagnetic fields with inertial bodies. Constitutive equations describing the coupling of multibody dynamics with Kirchhoff's theory define discrete electromechanical systems. Based on the principle of virtual work the motion equations are Lagrange's equations of second kind. The automatic generation of these model equations based on a unique approach is presented.

1 Introduction

Electromechanical systems (EMS) are physical structures characterized by interaction between electromagnetic fields and inertial bodies [4]. The interaction can be expressed by constitutive equations (force law) describing the coupling of Maxwell's theory and mechanics. Constitutive equations describing the coupling between the dynamics of multibody systems (MBS) with a finite degree of freedom and Kirchhoff's theory (as quasi stationary approximation of Maxwell's theory) define discrete EMS. A mathematical description oriented by classical analytical mechanics and completed by some basic concepts and methods of graph theory to characterize topological properties of electrical networks plays a fundamental role for a unique modelling and simulation of discrete EMS. In view of such a unique approach, concepts, definitions and notations of analytical multibody dynamics will be used [1]. That is justified because on the one hand analytical multibody dynamics can be regarded as a possible special case of EMS and on the other hand its concepts and definitions are developed very well to describe other physical structures in the same manner.

2 Electrical Systems

Using some basic concepts and methods of multibody dynamics [2], the electrical system dynamics can be described in the same manner very practically. The application of Lagrange's approach to electrical systems (ES) with lumped parameters is based on the concept of a multipole and its representation by abstract 2-poles as well as Kirchhoff's theory and the principle of virtual work. The *internal* dynamical behaviour of electrical components (resistor, capacitor, inductor etc.) can be represented by external measurements at the terminals using voltmeters and ammeters. This leads to a model of a construction element by using a *black box* endowed with a finite number of terminals. A black box having two terminals to measure voltage and current according to figure 1 is called an *electrical 2-pole* (e.g. a coil, capacitor etc.). The terminals of the voltmeter/ammeter have "+" and "−" symbols. Positive (negative) indication means voltage $V(t) \underset{(<)}{>} 0$ and current $I(t) \underset{(<)}{>} 0$. The conditions how to connect volt-/ammeters are defined by a *reference arrow*, so that the initial point of which corresponds to "+" terminal and the final point corresponds to "−" terminal. This way, a "coordinate system" (in the sense of mechanics) is defined at each 2-pole. The 2-pole-element defines a branch of the ES. The measured quantities $V(t)$ and $I(t)$ are called *branch voltage* and *branch current.* Their mathematical relationship in general given by a nonlinear operator equation is called a *branch relation* (constitutive equation).

Definition 1 *An electrical n-pole is defined by a black box with n, $2 \leq n < \infty$, terminals and the following properties:*

a) *Voltages and currents can be measured between any two terminals at the same time;*
b) *Kirchhoff's laws must be satisfied related to the measured voltages and currents.*

If the measurement results are independent on external influences then the black box is called *proper n-pole* else *pseudo n-pole*. The electrical properties of an n-pole can be characterized by the set of all relations (determined by external measurements) between $\binom{n}{2}$ voltages at any two terminals and the n

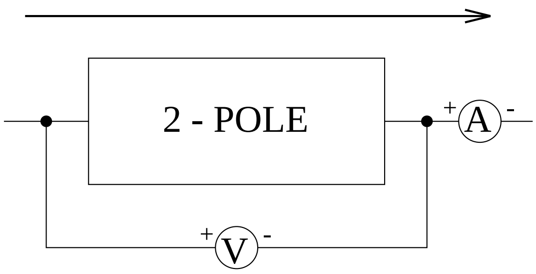

Figure 1:
Measurement instructions at 2-pole

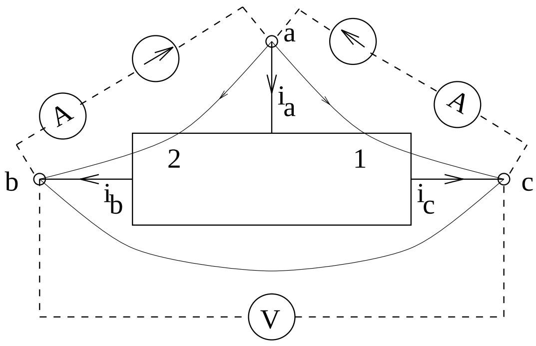

Figure 2:
External measurements at 3-pole
$i_a = I^1 + I^2$, $i_b = I^2$, $i_c = I^1$

currents through the terminals (figure 2). The $\binom{n}{2}$ external measurements of voltages at an n-pole define a so-called *complete measurement graph* Γ_M. Its nodes represent the terminals of the n-pole, and its branches represent the *fictitious 2-poles*, the voltages of which are measured. Prescribing the voltages in an arbitrary frame $G(\Gamma_M)$, the other voltages in Γ_M are uniquely determined due to Kirchhoff's voltage law. The currents measured in the branches of the frame $G(\Gamma_M)$ determine uniquely the currents through the terminals of the n-pole. Hence, the *electrical properties* of an n-pole can be characterized by (measured) relations between the $n-1$ voltages and the $n-1$ currents in $G(\Gamma_M)$.

Definition 2 *A complete measurement graph Γ_M with exactly two nodes and one relation is called an* abstract 2-pole *if it contains at least one of the both variables V, I of the measurement graph.*

2.1 Kirchhoff's Theory of Electrical Systems

Definition 3 *An electrical system (ES) is a finite set of galvanically connected electrical multipoles (finite multipole network).*

After defining all pole-graphs, an ES is represented by a finite network of abstract 2-poles with the network graph Γ (containing B branches, N nodes, p components). Let G denote an arbitrary (but fixed) frame of Γ and $H(G)$ the coframe of G in Γ. The corresponding fundamental cut and the fundamental loop matrices are denoted by Q and A, respectively.

Then, *Kirchhoff's laws* become:

$$\sum_{i\in\Gamma} A^i{}_j V_i = 0, \quad j \in H \quad (\text{"voltage law"}), \tag{2.1}$$

$$\sum_{i\in\Gamma} Q_i{}^j I^i = 0, \quad j \in G \quad (\text{"current law"}). \tag{2.2}$$

(2.1) describes the vanishing of the sum of all (oriented) voltages across each fundamental loop $j \in H$. (2.2) describes the vanishing of the sum of all (oriented) currents across each fundamental cut $j \in G$. (Symbols of subgraphs are used simultaneously to denote corresponding index sets.) Both linear equation systems have full rank ($rank(A) = B - N + p$, $rank(Q) = N - p$) and are equivalent to Kirchhoff's voltage law or Kirchhoff's current law, respectively ($b^i{}_j V_i = 0$, $a^i{}_j I^j = 0$; $b^i{}_j$: mesh-branch-inzidence-matrix, $a^i{}_j$: node-branch-inzidence-matrix). Together with the B branch relations of the abstract 2-poles (constitutive equations), (2.1) and (2.2) constitute a system of $2B$ equations in order to determine the $2B$ functions $V_i(t)$, $I^i(t)$, $i \in \Gamma$.

A and Q are related to each other by

$$A^i{}_j = \begin{cases} \delta^i{}_j \ , & i, j \in H \\ -Q_j{}^i \ , & i \in G,\ j \in H. \end{cases} \tag{2.3}$$

Hence, $A^T Q = 0$.

2.2 Lagrange's Equations for ES

The Principle of Virtual Work

(2.2) yields

$$I^j = -Q_i{}^j I^i, \quad i \in H, \, j \in G,$$

and due to (2.3) holds

$$I^j = A^j{}_\mu i^\mu, \quad \mu \in H, \, j \in \Gamma. \tag{2.4}$$

That means, each current I^j, $j \in \Gamma$, can be represented by a linear combination of coframe currents i^μ, $\mu \in H$. (2.4) is called *mesh-transformation* (MT), it defines the kinematics of the ES in the so-called charge formulation.

Generalized coordinates of ES are introduced as follows:
Defining functions

$$\bar{q}^j(t) := \int_0^t I^j(\tau)\, d\tau, \quad \bar{\lambda}_j(t) := \int_0^t V_j(\tau)\, d\tau \tag{2.5}$$

as *charge* and *flux* of the abstract 2-pole j, $j \in \Gamma$, respectively, its branch relation has the form

$$\dot{\bar{\lambda}}_j \equiv V_j = f_j(\ddot{\bar{q}}, \dot{\bar{q}}, \bar{q}, t) \quad \text{or} \quad \bar{\lambda}_j = f_j(\dot{\bar{q}}, \bar{q}, t) \quad \text{or} \quad \bar{q}^j = g_0^j(t), \tag{2.6}$$

where f_j and g_0^j are given, differentiable functions (notation: $\bar{q} := (\bar{q}^1, \ldots, \bar{q}^B)$, $\bar{\lambda} := (\bar{\lambda}_1, \ldots, \bar{\lambda}_B)$). $\bar{q}^j$ denotes the charge which has been moved in (0,t) caused by the current I^j; $\bar{\lambda}_j$ has not necessarily the meaning of "magnetic flux". An abstract 2-pole with the relation $\bar{q}^j = g_0^j(t)$ is called *current generator*, $g_0^j(t)$ denotes its charge source. Hence, the equations (2.6) are the constitutive equations of an ES (in the sense of mechanics).

Due to the mesh-transformation (2.4), a frame G must not contain current generators. Consequently, all current generators belong to the coframe $H(G)$ and decomposite the index set

$$\begin{aligned} H(G) \quad &= \quad H^* \cup H_0\,, \\ &\quad H^* \;:\; \text{coframe branches not containing current generators,} \\ &\quad H_0 \;:\; \text{coframe branches containing current generators,} \end{aligned} \tag{2.7}$$

so that (2.4) with (2.5) yields

$$\bar{q}^j = A^j{}_\mu q^\mu + q_0^j(t), \quad q_0^j := A^j{}_\lambda \bar{q}^\lambda \equiv A^j{}_\lambda g_0^\lambda(t), \quad j \in \Gamma, \, \mu \in H^*, \, \lambda \in H_0. \tag{2.8}$$

The set of branch charges $\{\bar{q}^j, j \in \Gamma\}$ is called a *configuration* of the ES. All charges $\bar{q}$ fulfilling Kirchhoff's current law in integrated form $a^i{}_j \bar{q}^j = 0$ at time t determine the set

$$\mathfrak{L}_t := \{\bar{q}^j \,|\, \bar{q}^j = A^j{}_\mu q^\mu + q_0^j(t);\; q^\mu \in \mathbb{R}\} \tag{2.9}$$

of all *admissible configurations* of the ES at time t.

In the following, latin indices are related to Γ, greek indices to H^*, and the well-known summation convention will be used. Denoting the power of H^* by m, (2.9) defines a 1-1 map of $\mathfrak{L}_t$ to $\mathbb{R}^m$.

Definition 4 *The ES is said to be* holonomic *having the* quasi degree of freedom m. $q = (q^\mu, \mu \in H^*)$ *is called its* representing point, $q \in \mathbb{R}^m$*;* $\mathbb{R}^m$ *is called its* configuration space*, the* q^μ, $\mu \in H^*$, *are called (topologically generated)* generalized coordinates *of the ES.*

The motion of the ES is described by C^2-functions $q = q(t)$. The state of the ES is given by $(\dot{q}, q)$. The ES is called *scleronomic* if $\mathfrak{L}_t$ does not depend on t explicitly, otherwise it is called *rheonomic* ($H_0 \neq \emptyset$).

A *virtual displacement* of an ES is defined by a set of differential increments of charges $\bar{q}^j$ belonging to a variation δq^μ, $\mu \in H^*$, at fixed time t:

$$\{\delta\bar{q}^j \,|\, \delta\bar{q}^j = A^j{}_\mu \delta q^\mu, \quad j \in \Gamma,\, \delta q^\mu \text{ arbitrary}\}. \tag{2.10}$$

(Because $A^j{}_\mu = \delta^j_\mu$, $j, \mu \in H$, it is $\delta\bar{q}^j = 0 \quad \forall j \in H_0$.) Motion and state of an ES are defined analogously to mechanical systems [2].

Axiom 1 *(Principle of virtual work in view of the charge approach)*
Let the constitutive equations be given as

$$\dot{\lambda}_i \equiv V_i = f_i(\ddot{\bar{q}}, \dot{\bar{q}}, \bar{q}, t), \quad i \in \Gamma \setminus H_0.$$

Then

$$\delta' A := -V_i \delta\bar{q}^i = 0 \quad \forall \delta\bar{q}^i \textit{ virtual} \tag{2.11}$$

defines the actual motion of an ES.

Assuming that the constitutive equations satisfy certain necessary and sufficient conditions with respect to the existence of a kinetic potential of first order (Helmholtz-conditions), the axiom 1 yields the Lagrange equations of motion of the ES

$$(\dot{\partial}_\mu \Lambda)^{\cdot} - \partial_\mu \Lambda + \dot{\partial}_\mu D = Q^{(S)}_\mu, \tag{2.12}$$

where $\Lambda(\dot{q}, q, t) := W'_m - W_e - V^h$ is the Lagrangian (W'_m - magnetic coenergy, W_e - electric energy, V^h - generalized potential) and $D(\dot{q}, q, t) := D^{(0)} + D^{(1)}$ denotes the dissipation function of the ES. $Q^{(S)}_\mu := -A^i{}_\mu V^{(S)}_i|_{MT}$ are those voltages in the fundamental circuit μ that cannot or should not be represented by Λ or D.

A Lagrange Model for ES

Let the constitutive equations of an ES be given by

$$\begin{aligned}
V^{(L)}_i &= \dot{\Psi}_i, \quad \Psi_i(\dot{\bar{q}}, t) = L_{ij}(t)\dot{\bar{q}}^j + \Psi_{i0}(t), && \\
V^{(R)}_i &= R_{ij}(t)\dot{\bar{q}}^j && \text{(rheolinear resistors)}, \\
V^{(C)}_i &= C_{ij}(t)\bar{q}^j + V^{(C)}_{i0}, && \\
V_{i0} &= V_{i0}(t) && \text{(voltage generators)}
\end{aligned}$$

with

$$L_{ij} = L_{ji}, \quad C_{ij} = C_{ji}, \quad \partial_0 R_{[ij]} \equiv 0, \tag{2.13}$$

where $\Psi_{i0}(t)$, $V_{i0}(t)$, $V^{(C)}_{i0}$ (initial voltage of a capacitor) are arbitrary and $\bar{q}^i(t) := g^i_0(t)$, $i \in H_0$.

Hence, the magnetic copotential reads

$$\begin{aligned}
W'_m(\dot{q}, t) &:= \frac{1}{2} A^i{}_\mu A^j{}_\nu L_{ij} \dot{q}^\mu \dot{q}^\nu + A^i{}_\mu [L_{ij} \dot{q}^j_0(t) + \Psi_{i0}(t)] \dot{q}^\mu, \\
& \quad\; q^j_0(t) := A^j{}_\lambda g^\lambda_0(t),
\end{aligned} \tag{2.14}$$

and the electric potential reads

$$W_e(q, t) := \frac{1}{2} A^i{}_\mu A^j{}_\nu C_{ij} q^\mu q^\nu + A^i{}_\mu [C_{ij} q^j_0(t) + V_{i0}(t) + V^{(C)}_{i0}] q^\mu. \tag{2.15}$$

The dissipation function

$$D^{(1)}(\dot{q},t) := \frac{1}{2} A^i{}_\mu A^j{}_\nu R_{(ij)} \dot{q}^\mu \dot{q}^\nu + A^i{}_\mu R_{ij} \dot{q}_0^j(t) \dot{q}^\mu \tag{2.16}$$

and the generalized (gyroscopic) potential

$$V(\dot{q},q) := \frac{1}{2} A^i{}_\mu A^j{}_\nu R_{[ij]} q^\mu \dot{q}^\nu \equiv \frac{1}{2} r_{[\mu\nu]} (q^\mu \dot{q}^\nu - q^\nu \dot{q}^\mu)|_{\mu<\nu} \tag{2.17}$$

can be assigned to rheolinear resistors. Hence, the Lagrangian model $\{\Lambda, D\}$ of the ES is given by

$$\begin{aligned} \Lambda(\dot{q},q,t) &:= W_m' - W_e - V \equiv \tfrac{1}{2} A^i{}_\mu A^j{}_\nu [L_{ij}(t)\dot{q}^\mu \dot{q}^\nu - C_{ij}(t) q^\mu q^\nu - R_{[ij]} q^\mu \dot{q}^\nu] + \\ &\quad + A^i{}_\mu \{[L_{ij}(t)\dot{q}_0^j(t) + \Psi_{i0}(t)]\dot{q}^\mu - [C_{ij}(t) q_0^j(t) + V_{i0}(t) + V_{i0}^{(C)}] q^\mu\} \\ D(\dot{q},t) &:= \tfrac{1}{2} A^i{}_\mu A^j{}_\nu R_{(ij)} \dot{q}^\mu \dot{q}^\nu + A^i{}_\mu R_{ij} \dot{q}_0^j(t) \dot{q}^\mu. \end{aligned}$$

Both functions are quadratic in $\dot{q}$:

$$\Lambda = \frac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu + g_{\mu 0} \dot{q}^\mu + \frac{1}{2} g_{00}, \quad D = \frac{1}{2} r_{(\mu\nu)} \dot{q}^\mu \dot{q}^\nu + r_\mu \dot{q}^\mu$$

with

$$\begin{aligned} g_{\mu\nu}(t) &:= l_{\mu\nu}(t) \equiv A^i{}_\mu A^j{}_\nu L_{ij}(t), \\ g_{\mu 0}(q,t) &:= A^i{}_\mu [L_{ij}(t)\dot{q}_0^j(t) + \Psi_{i0}(t) + \tfrac{1}{2} A^j{}_\nu R_{[ji]} q^\nu], \\ g_{00}(q,t) &:= -A^i{}_\mu \{C_{ij}(t)[A^j{}_\nu q^\nu + 2q_0^j(t)] + 2[V_{i0}(t) + V_{i0}^{(C)}]\} q^\mu, \\ r_{(\mu\nu)}(t) &:= A^i{}_\mu A^j{}_\nu R_{(ij)}(t), \qquad r_\mu(t) := A^i{}_\mu R_{ij} \dot{q}_0^j(t). \end{aligned}$$

Then, the Lagrange's equations can be written in their explicit form

$$g_{\mu\nu} \ddot{q}^\nu + \Gamma_{\mu\alpha\beta} \dot{q}^\alpha \dot{q}^\beta + r_{(\mu\nu)} \dot{q}^\nu + r_\mu = 0$$

($\mu, \nu \in H^*$; $\alpha, \beta \in H^* \cup \{0\}$; $q^0 = t$; $\dot{q}^0 = 1$) with

$$\begin{aligned} \Gamma_{\mu\nu\varrho} & & &\equiv 0 \text{ for } \nu, \varrho \in H^*, \\ \Gamma_{\mu\varrho 0} &= \Gamma_{\mu 0 \varrho} & &= \tfrac{1}{2} A^i{}_\mu A^j{}_\varrho \dot{L}_{ij} + \tfrac{1}{4} A^i{}_\mu A^j{}_\varrho R_{[ij]}, \\ \Gamma_{\mu 00} & & &:= \partial_0 g_{\mu 0} - \tfrac{1}{2} \partial_\mu g_{00}. \end{aligned}$$

3 Discrete Electromechanical Systems

The Lagrange formalism is described for electromechanical systems (EMS) [3]. This will be done based on both, analytical mechanics (i.e. its application in multibody dynamics) and the formalism described for electrical systems in section 2. EMS can be regarded as physical structures characterized by interactions of electromagnetic fields with inertial bodies. The interaction is expressed by constitutive equations describing the coupling between Maxwell's theory and mechanics. Constitutive equations, describing the coupling of multibody dynamics with Kirchhoff's theory (as quasi stationary approximation of Maxwell's theory), define discrete EMS. In the following, only such systems will be regarded.

Definition 5 *An EMS is a finite set of physical objects with mechanical and/or electrical multipole-properties interacting among themselves by electrodynamical and/or electro-magneto-mechanical coupling: multibody dynamics* ∪ *Kirchhoff's theory.*

3.1 Kinematics of Electromechanical Systems

The kinematics of an EMS is defined by the geometric constraints between the rigid bodies and the topology of the electrical network represented by abstract 2-poles. The set $\{\bar{q}^i,\, \underset{k}{\bar{x}}^s |\ i \in \Gamma,\ s = 1, \ldots, 6;\ k = 1, \ldots, K\}$ is called a *configuration* of the EMS, where the $\bar{q}^i$ denote the branch charges (Γ = graph of the 2-pole-network) and the $\underset{k}{\bar{x}}^s$ denote the mechanical coordinates. All branch charges $\bar{q}^i$ and coordinates $\underset{k}{\bar{x}}^s$, which fulfil all constraints of the EMS at given time t (i.e. geometric constraints and Kirchhoff's current law), determine the set

$$\mathfrak{L}_t := \left\{ \bar{q}^i, \underset{k}{\bar{x}}^s | \bar{q}^i = A^i{}_\mu q^\mu + q_0^i(t), \underset{k}{\bar{x}}^s = \underset{k}{\bar{x}}^s(q^\kappa, t); \begin{array}{c} i \in \Gamma, s = 1, ..., 6; k = 1, ..., K; \\ (q^\mu, q^\kappa) \in \mathbb{R}^n, \mu \in H^*, \kappa \in J \end{array} \right\} \tag{3.1}$$

of all *admissible configurations* of an EMS at time t. H^* denotes the subset of the coframe $H(G)$ of Γ not containing current generators, J is an index set of mechanical coordinates q^κ; $J \cap H^* = \emptyset$. Convention: In the following, $\mu, \nu, \omega \in H^*$, $\kappa, \lambda, \rho, \in J$, $a, b, c \in H^* \cup J$, $\alpha, \beta \in H^* \cup J \cup \{0\}$, and $q^0 = t$, $\dot{q}^0 = 1$ have to be assumed. $\mathfrak{L}_t$ is a 1-1 map to a cylindrical domain $D \subset \mathbb{R}^n$ of the configuration space, $n := |H^*| + |J|$. n is called quasi degree of freedom of the EMS, and $q = (q^a) = (q^\mu, q^\kappa)$ denotes its representing point. The mesh charges q^μ and the mechanical coordinates q^κ are the generalized coordinates of the EMS. The motion of the EMS is given by C^2-functions $q = q(t)$, and the state of the EMS is given by $(\dot{q},\, q)$. The EMS is called *holonomic* if there are no nonintegrable kinematic constraints, otherwise it is called *anholonomic*. The EMS is called *scleronomic* if $\mathfrak{L}_t$ does not depend on t explicitly, otherwise it is called *rheonomic*.

3.2 Constitutive Equations

Let $d\mathfrak{k}(\xi)$ denote the applied forces acting on the bodies of the EMS, V_i denote the voltages of abstract 2-poles, and Q_κ and v_μ the generalized forces and the mesh voltages, respectively. Hence,

$$Q_\kappa := \mathcal{S} \partial_\kappa \mathfrak{x} d\mathfrak{k}, \quad v_\mu := A^i{}_\mu\ V_i|_{MT}\,. \tag{3.2}$$

Assumtion:
The constitutive equations of an EMS

$$Q_\kappa = Q_\kappa(\dot{q}^a, q^a, t), \quad v_\mu = v_\mu(\ddot{q}^a, \dot{q}^a, q^a, t) \tag{3.3}$$

are given by sufficient smooth functions Q_κ and v_μ.

The simultaneous presence of q^λ and q^ν and their derivatives in these equations indicates the electromechanical interaction. The description of Q_κ and v_μ by state functions $\Omega(\dot{q}, q, t)$ and $D(\dot{q}, q, t)$ is necessary for a unique representation of the motion equations in Lagrange's notation:

$$Q_a^* \equiv \delta_a \Omega - \dot{\partial}_a D \tag{3.4}$$

with

$$Q_a^* := \left\{ \begin{array}{rl} -v_a\ , & a \in H^* \\ Q_a\ , & a \in J. \end{array} \right. \tag{3.5}$$

Hence, the structure of the functions Q_a^* and Ω follows (not considering any physical background):

a) $v_\mu = \dot{\psi}_\mu + u_\mu$ with $\psi_\mu = \psi_\mu(\dot{q}^\nu, q^a, t)$, $\quad \dot{\partial}_{[\nu} \psi_{\mu]} = 0$ and $u_\mu = u_\mu(\dot{q}, q, t)$,

b) $\Omega = -\Psi + V$ with $\Psi(\dot{q}^\nu, q^a, t) := \int \psi_\mu d\dot{q}^\mu$, $\quad V(\dot{q}, q, t) := \omega_a(q, t)\dot{q}^a + \omega_0 \equiv \omega_\alpha \dot{q}^\alpha$,

c) $Q_a := \partial_a \Psi + \delta_a V - \dot{\partial}_a D$ with $Q_a := \left\{ \begin{array}{rl} -u_\mu\ , & a = \mu \in H^* \\ Q_\kappa\ , & a = \kappa \in J \end{array} \right.$ and $\delta_a V \equiv 2\partial_{[\alpha} \omega_{a]} \dot{q}^\alpha$.

The partitioning

$$\begin{array}{rclcccccc} Q_\kappa & = & & & Q_\kappa^{(0)} & + & Q_\kappa^{(1)} & + & Q_\kappa^{(2)}, \\ -v_\mu & = & - & \dot\psi_\mu & - \; u_\mu^{(0)} & - & u_\mu^{(1)} & - & u_\mu^{(2)} \end{array} \tag{3.6}$$

with

$$\begin{array}{llll} \psi_\mu = \dot\partial_\mu \Psi, & u_\mu^{(0)} = -\partial_\mu \Psi, & Q_\kappa^{(0)} = \partial_\kappa \Psi, & (Q_a^{(0)} = \partial_a \Psi), \\ & u_\mu^{(1)} = -\delta_\mu V, & Q_\kappa^{(1)} = \delta_\kappa V, & (Q_a^{(1)} = \delta_a V), \\ & u_\mu^{(2)} = \dot\partial_\mu D, & Q_\kappa^{(2)} = -\dot\partial_\kappa D, & (Q_a^{(2)} = -\dot\partial_a D) \end{array} \tag{3.7}$$

is a sufficient condition to represent Q_κ and v_μ by state functions Ω, D due to (3.4).

3.3 State Functions

The arbitraryly given state functions Ψ, V, D describe the classes K^0, K^1 and K^2 according to (3.7) . On the other hand, if the functions Q_a are given by (3.3) and if there is a decomposition $Q_a = Q_a^{(0)} + Q_a^{(1)} + Q_a^{(2)}$ fulfilling the integrability conditions belonging to (3.7), the state functions can be calculated:

the magnetomechanical copotential

$$\Psi := \int_{(0,0,0)}^{(\dot q^\mu, q^\mu, q^\kappa)} \psi_\mu d\dot q^\mu - u_\mu^{(0)} dq^\mu + Q_\kappa^{(0)} dq^\kappa; \tag{3.8}$$

the generalized electromechanical potential

$$V = \omega_\alpha(q,t)\dot q^\alpha \equiv \omega_a \dot q^a + \omega_0, \tag{3.9}$$

ω_a, ω_0 defined by a PDE-system [3];

the dissipation function

$$D := \int_{(0,0)}^{(\dot q^\mu, \dot q^\kappa)} u_\mu^{(2)} d\dot q^\mu - Q_\kappa^{(2)} d\dot q^\kappa. \tag{3.10}$$

3.4 Kinetics of Electromechanical Systems

The kinetics of EMS is based on the principle of virtual work in Lagrange's notation.

Axiom 2 *The actual motion of an EMS is characterized by the vanishing of the virtual work*

$$\delta' A := -V_i \delta \bar q^i + \mathcal{S} \delta \mathfrak{x} (d\mathfrak{k} - \ddot{\mathfrak{x}} dm) = 0 \quad \forall\, \delta \bar q^i, \delta \mathfrak{x} \textit{ virtual} \tag{3.11}$$

at any time t.

($\mathcal{S}$: Summation over all material points ξ of the EMS, $d\mathfrak{k}$: applied forces arbitraryly distributed, dm: mass element, $\dot{\mathfrak{x}} = \frac{d}{dt}\mathfrak{x}(\xi, q^\kappa, t)$: velocity, $\ddot{\mathfrak{x}} = \frac{d}{dt}\dot{\mathfrak{x}}$: acceleration of ξ related to an inertial frame, V_i: voltages of abstract 2-poles of the representing graph Γ of the electrical network, $\bar q^i$: branch charges of Γ.)

Using the kinetic energy of the EMS

$$T(\dot q^\lambda, q^\lambda, t) := \frac{1}{2} \mathcal{S} \dot{\mathfrak{x}}^2 dm$$

and the generalized forces Q_κ and mesh voltages v_μ

$$\mathcal{S} \delta \mathfrak{x} d\mathfrak{k} \equiv \mathcal{S} \partial_\kappa \mathfrak{x} d\mathfrak{k} \delta q^\kappa \equiv Q_\kappa \delta q^\kappa, \quad V_i \delta \bar q^i \equiv V_i A^i{}_\mu \delta q^\mu \equiv v_\mu \delta q^\mu,$$

(3.11) yields

$$\delta' A = -v_\mu \delta q^\mu + [-(\dot\partial_\kappa T)^{\cdot} + \partial_\kappa T + Q_\kappa]\delta q^\kappa = 0 \quad \forall\, \delta q^\mu,\, \delta q^\kappa.$$

The motion equations of the EMS

$$v_\mu = 0, \quad (\dot\partial_\kappa T)^{\cdot} - \partial_\kappa T = Q_\kappa \tag{3.12}$$

are Kirchhoff's mesh equations and "mechanical" Lagrange's equations. With respect to (3.4) and $\partial_\mu T \equiv 0$, $\dot\partial_\mu T \equiv 0$, they read in Lagrange's form

$$\begin{aligned} (\dot\partial_a \Lambda)^{\cdot} - \partial_a \Lambda + \dot\partial_a D &= 0, \quad a \in J \cup H^*, && (3.13)\\ \Lambda := T - \Omega &= T(\dot q^\lambda, q^\lambda, t) + \Psi(\dot q^\nu, q^a, t) - V(\dot q, q, t). && (3.14) \end{aligned}$$

If v_μ, Q_κ are given, Ψ, V und D can be calculated according to (3.8), (3.9) and (3.10). Parts of the generalized forces $v_\mu^{(S)}$, $Q_\kappa^{(S)}$, which cannot or should not be represented by these state functions, have to be taken into account on the right-hand side of (3.13):

$$(\dot\partial_a \Lambda)^{\cdot} - \partial_a \Lambda + \dot\partial_a D = Q_a^{(S)}, \tag{3.15}$$

$$Q_a^{(S)} := \begin{cases} -v_\mu^{(S)}, & a = \mu \in H^* \\ Q_\kappa^{(S)}, & a = \kappa \in J. \end{cases}$$

3.5 The Structure of Lagrange's Equations of Motion

Starting from the Lagrange model $\{\Lambda, D\}$

$$\Lambda = \tfrac{1}{2} g_{\alpha\beta}(q,t)\dot q^\alpha \dot q^\beta, \quad D = \tfrac{1}{2} s_{\alpha\beta}(q,t)\, \dot q^\alpha \dot q^\beta$$

with

$$\begin{aligned} (\dot\partial_a \Lambda)^{\cdot} &= (g_{a\beta}\dot q^\beta)^{\cdot} = g_{ab}\ddot q^b + \partial_\alpha g_{a\beta}\dot q^\alpha \dot q^\beta,\\ \partial_a \Lambda &= \tfrac{1}{2}\partial_a g_{\alpha\beta}\dot q^\alpha \dot q^\beta,\\ \dot\partial_a D &= s_{a\beta}\dot q^\beta \end{aligned}$$

and

$$\Gamma_{a\alpha\beta} := \frac{1}{2}(\partial_\alpha g_{a\beta} + \partial_\beta g_{a\alpha} - \partial_a g_{\alpha\beta}),$$

the Lagrange's equations in explicit form read:

$$g_{ab}(q,t)\ddot q^b + \Gamma_{a\alpha\beta}(q,t)\dot q^\alpha \dot q^\beta + s_{a\beta}\dot q^\beta = 0. \tag{3.16}$$

Here is

$$g_{ab} := \dot\partial_a \dot\partial_b \Lambda = g_{ba},$$

and with $\Lambda = T + \Psi - V$ follows

$$(g_{ab}) = \begin{bmatrix} g_{\mu\nu} & 0 \\ 0 & g_{\kappa\lambda} \end{bmatrix}, \begin{aligned} g_{\mu\nu} = \dot\partial_\mu \dot\partial_\nu \Psi = l_{\mu\nu} &= A^i{}_\mu A^j{}_\nu L_{ij},\\ g_{\kappa\lambda} = \dot\partial_\kappa \dot\partial_\lambda T = \sum_k [\underset{k}{m}\underset{k}{u}_{i\kappa}\underset{k}{u}_{i\lambda} &+ (1 - s_\kappa)(1 - s_\lambda)\underset{k}{\Theta}^{ij}\underset{k}{\Omega}_{i\kappa}\underset{k}{\Omega}_{j\lambda}]. \end{aligned}$$

Hence, g_{ab} is the direct sum of the two matrices $g_{\mu\nu}$ and $g_{\kappa\lambda}$. Because of the quasi stationary approximation of Maxwell's theory ($\partial_\mu \Psi \equiv 0$), g_{ab} is independent of q^μ. If $g_{\mu\nu}$ is a regular matrix, g_{ab} is regular, too ($g_{\kappa\lambda}$ is regular because of the positive definiteness of the kinetic energy T). Assuming $g_{\mu\nu}$ is regular, $g_{ab}(q,t)$ defines (in general) a time-dependent Riemannian metric in $\mathbb{R}^n$. The $\Gamma_{abc}(q,t)$ are the time-dependent Christoffel symbols of first kind.

Using

$$\begin{aligned} &\Gamma_{\mu\nu\lambda} = \tfrac{1}{2}\partial_\lambda g_{\mu\nu} = \tfrac{1}{2}\partial_\lambda l_{\mu\nu}, && \Gamma_{\kappa\nu\omega} = -\tfrac{1}{2}\partial_\kappa g_{\nu\omega} = -\tfrac{1}{2}\partial_\kappa l_{\nu\omega},\\ &\Gamma_{\mu\lambda\varrho} \equiv 0, \quad \Gamma_{\mu\nu\omega} \equiv 0, \quad \Gamma_{\kappa\lambda\nu} \equiv 0, && \mu,\nu,\omega \in H^*; \kappa,\lambda,\varrho \in J, \end{aligned}$$

(3.16) yields

$$
\boxed{\begin{array}{llllllll}
g_{\mu\nu}\ddot{q}^{\nu} & & +\partial_{\lambda}g_{\mu\nu}\dot{q}^{\lambda}\dot{q}^{\nu} & +(2\Gamma_{\mu b0}+s_{\mu b})\dot{q}^{b} & +\Gamma_{\mu 00} & +s_{\mu 0} & =0 \\
g_{\kappa\lambda}\ddot{q}^{\lambda} & +\Gamma_{\kappa\lambda\varrho}\dot{q}^{\lambda}\dot{q}^{\varrho} & -\frac{1}{2}\partial_{\kappa}g_{\nu\omega}\dot{q}^{\nu}\dot{q}^{\omega} & +(2\Gamma_{\kappa b0}+s_{\kappa b})\dot{q}^{b} & +\Gamma_{\kappa 00} & +s_{\kappa 0} & =0.
\end{array}} \tag{3.17}
$$

This structure of Lagrange's equations follows from (3.16) with respect to a partitioning of the generalized coordinates $q = (q^{\mu}, q^{\kappa})$ and the special representation of the matrix g_{ab} as direct sum of $g_{\mu\nu}$ and $g_{\kappa\lambda}$ due to the quasi stationary approximation of Maxwell's theory.

The Christoffel symbols appearing in (3.17) can be calculated as:

$$
\begin{aligned}
\Gamma_{\kappa\lambda\varrho} &= (1-s_{\lambda})\sum_{k=\varrho}^{K}\varepsilon_{i}{}^{qr}[\underset{k}{m}\,\delta^{ij}\underset{k}{u}_{j\kappa}\underset{k}{\Omega}_{q\lambda}\underset{k}{u}_{r\varrho}+(1-s_{\kappa})(1-s_{\varrho})\underset{k}{\vartheta}^{ij}\underset{k}{\Omega}_{r\kappa}\underset{k}{\Omega}_{j\lambda}\underset{k}{\Omega}_{q\varrho}], \\
\Gamma_{\mu\nu 0} &= \frac{1}{2}\partial_{0}l_{\mu\nu}-\partial_{[\nu}\omega_{\mu]}, \\
\Gamma_{\mu\lambda 0} &= \frac{1}{2}\partial_{\lambda}(\psi_{\mu 0}-\omega_{\mu}), \\
\Gamma_{\mu 00} &= \partial_{0}\psi_{\mu 0}+\partial_{\mu}\omega_{0}, \\
\Gamma_{\kappa\nu 0} &= -\Gamma_{\nu\kappa 0}, \\
\Gamma_{\kappa\lambda 0} &= \frac{1}{2}\partial_{0}\dot{\partial}_{\kappa}\dot{\partial}_{\lambda}T_{2}-\partial_{[\lambda}\dot{\partial}_{\kappa]}T_{1}, \\
\Gamma_{\kappa 00} &= \partial_{0}\dot{\partial}_{\kappa}T_{1}-\partial_{\kappa}(T_{0}+\Psi_{0}-V_{0}).
\end{aligned}
$$

Some terms in (3.17) have a simple physical meaning. $g_{\mu\nu}\ddot{q}^{\nu}$ describes induced voltages as a result of changing currents and $\partial_{\lambda}g_{\mu\nu}\dot{q}^{\lambda}\dot{q}^{\nu}$ is the Coriolis voltage. The term $\frac{1}{2}\partial_{\kappa}g_{\nu\omega}\dot{q}^{\nu}\dot{q}^{\omega}$ describes generalized forces with electrical origin (Lorentz forces).

A special EMS (with inductivities, permanent magnets, resistors, current and voltage generators) with the constitutive equations

$$
\begin{aligned}
\Psi_{i} &= L_{ij}\dot{\bar{q}}^{j}+\Psi_{i0}, \qquad i,j\in H, \\
& \quad\; L_{ij}=L_{ij}(q^{\kappa}), \quad \Psi_{i0}=\Psi_{i0}(q^{\kappa}), \\
V^{(R)} &= R_{ij}\dot{\bar{q}}^{j}, \\
V_{i0} &= V_{i0}(t)
\end{aligned}
$$

and the force laws

$$
\begin{aligned}
\underset{k}{\mathfrak{k}} &= \underset{k}{K}^{(i)}\mathfrak{e}_{(i)}=\underset{r}{K}^{i}\underset{r}{\mathfrak{E}}_{i}, && \text{"mechanical" forces}, \\
Q_{\kappa} &= -\delta_{\kappa}(\Psi^{*}-V^{*}), && \text{"electromagnetic" forces},
\end{aligned}
$$

has Lagrange's equations of the following form:

$$
\begin{aligned}
& A^{i}{}_{\mu}A^{j}{}_{\nu}[L_{ij}\ddot{q}^{\nu}+\partial_{\lambda}L_{ij}\dot{q}^{\lambda}\dot{q}^{\nu}]+A^{i}{}_{\mu}\partial_{\kappa}[L_{ij}\dot{q}_{0}^{j}(t)+\Psi_{i0}]\dot{q}^{\kappa}+A^{i}{}_{\mu}V_{i0}(t) \\
& \qquad = -A^{i}{}_{\mu}A^{j}{}_{\nu}R_{ij}\dot{q}^{\nu}-A^{i}{}_{\mu}R_{ij}\dot{q}_{0}^{j}(t) \\
& g_{\kappa\lambda}\ddot{q}^{\lambda}+\Gamma_{\kappa\lambda\varrho}\dot{q}^{\lambda}\dot{q}^{\varrho}-\tfrac{1}{2}A^{i}{}_{\mu}A^{j}{}_{\nu}\partial_{\kappa}L_{ij}\dot{q}^{\mu}\dot{q}^{\nu}-A^{i}{}_{\mu}\partial_{\kappa}[L_{ij}\dot{q}_{0}^{j}(t)+\Psi_{i0}]\dot{q}^{\mu} \\
& \qquad = \sum_{k=\kappa}^{K}[\underset{k}{K}^{i}\underset{k}{u}_{i\kappa}+\underset{k}{M}^{i}\underset{k}{\Omega}_{i\kappa}]+\tfrac{1}{2}\partial_{\kappa}L_{ij}\dot{q}_{0}^{i}(t)\dot{q}_{0}^{j}(t)+\partial_{\kappa}\Psi_{i0}\dot{q}_{0}^{i}(t).
\end{aligned}
$$

4 Examples

Generalized Electric Machine

The unique Lagrange approach to dynamic simulation of discrete EMS will be shown shortly by the example of the generalized electric machine.

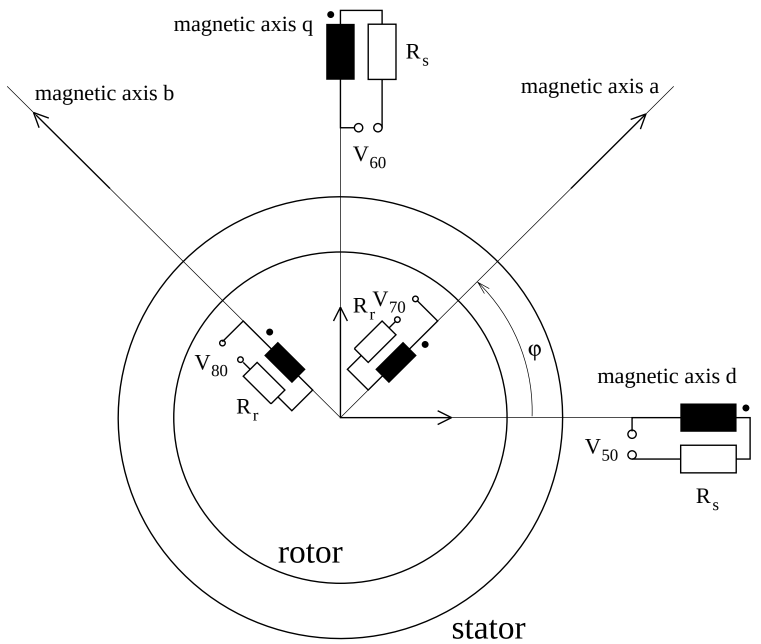

Figure 3: Generalized electric machine

An idealized 2-pole 2-phase-machine is called generalized electric machine (figure 3). The mechanical submodel consists of two bodies connected by a rotational joint. Both, rotor and stator should have two lumped inductances, displaced by $\pi/2$, which summarize all inductances of rotor and stator respectively. The quasi stationary approximation of Maxwell's theory is used. The magnetomechanical interaction is defined only by field distribution in the air gap. The magnetic material of rotor and stator should have a linear $\mathfrak{B}$-$\mathfrak{H}$-curve and no saturation. The structure graph consists of four fundamental loops, each of them containing an inductance, a voltage generator and a resistor (figure 4). The relative angle between rotor and stator is the only mechanical coordinate. The four charges in the fundamental loops of the structure graph are the generalized electrical coordinates.

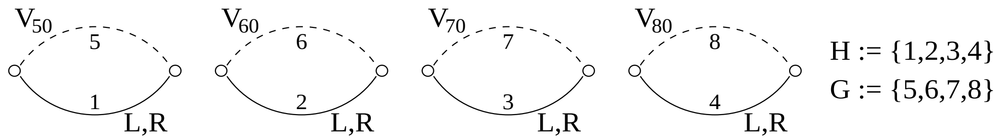

Figure 4: Graph of the generalized electric machine

The physical model yields the topology with the fundamental loop matrix

$$A^i{}_\mu = \delta^{i-4}_\mu, \quad i \in G,\, \mu \in H,\, H = H^*,\, H_0 = \emptyset,\, |H^*| = 4.$$

The generalized coordinates are denoted by

$$q \equiv (q^a) \equiv (q^\mu; q^\kappa) \equiv (q^{ds}, q^{qs}, q^{ar}, q^{br}; \varphi) \equiv (q^1, q^2, q^3, q^4; q^5)$$

(q^μ: charge in fundamental loop μ), and $\dot{q}^a$ are the generalized velocities of the electric machine. The pseudo degree of freedom is five.

The constitutive equations for a model of a smooth-air-gap machine read

$$\begin{array}{lcll}
\Psi_i & = & L_{ij} I^j, & L_{ij} = L_{ij}(\varphi) = L_{ji}; \\
M_{el} & = & \frac{1}{2} L'_{ij}(\varphi) I^i I^j, & L'(\varphi) = \frac{\partial L}{\partial \varphi}; \\
V_i^{(R)} & = & R_s I^i, & i = 1, 2, \quad R_s = const.; \\
V_i^{(R)} & = & R_r I^i, & i = 3, 4, \quad R_r = const.; \\
V_{i0} & = & V_{i0}(t), & i = 5, 6, 7, 8; \\
M_L & = & M_L(\dot{\varphi}, \varphi, t) & = \text{ loading torque, } M_r = -k\dot{\varphi} \quad (k > 0);
\end{array}$$

$$
(L_{ij}) = \begin{pmatrix} \bar{a} & 0 & \bar{c}\cos\varphi & -\bar{c}\sin\varphi \\ 0 & \bar{b} & \bar{d}\sin\varphi & \bar{d}\cos\varphi \\ \bar{c}\cos\varphi & \bar{d}\sin\varphi & e + f\cos 2\varphi & -f\sin 2\varphi \\ -\bar{c}\sin\varphi & \bar{d}\cos\varphi & -f\sin 2\varphi & e - f\cos 2\varphi \end{pmatrix}
$$

with the simplifying assumtions

$$
\bar{a} = \bar{b} = L_s, \quad \bar{c} = \bar{d} = M, \quad e = L_r, \quad f = 0
$$

(L_r, L_s - self-inductances of the rotor and the stator; M - mutual inductance).

Using this approach, the simulation tool **alaska** is able to generate automatically the complete system of motion equations of the generalized electric machine. In this case, these are five differential equations of second order. Hence, various investigations of the dynamics of different rotational electric machines, like induction or synchronous drives, even DC-machines, can be done. The unique Lagrange approach always guarantees the correct results with respect to generation of motion equations.

Other Applications

Some other applications, which have been investigated using the simulation tool **alaska**, should be shortly mentioned. In the case of the MAGLEV Transrapid, the main goal was to study some aspects of controlling electromagnets which are unstable in the opened loop. The carrying and tracking system is based on the attraction between the electromagnets in the vehicle and the ferromagnetic material in the track. It works contactlessly. This MAGLEV-model consists of five rigid bodies, the cabin and four suspension bodies. The cabin and the suspension bodies are coupled by springs. One magnet to carry the vehicle and one to track it are fixed at each suspension body. The drive is realized by stator windings in the track producing a travelling field (principle of linear synchronous motor). The parameters of the model (masses, stiffnesses etc.) are comparable to the technical data of the Transrapid TR06/TR07. The electrical network of this example consists of 28 branches (two branches for each magnet, six branches for drive windings on each side of the track). The degree of freedom of the mechanical subsystem is 30, the quasi degree of freedom of the complete system is 44. The control of the distance between the magnets and the track is implemented by control of the current of each magnet separately. PD-, PID- and state controls have been used in the entire model of the MAGLEV.

Another example is that of a planar motor driven directly. Some types of such drives with different constructions and working principles are investigated at our institute. One of the planar DC-motors, e.g., consists of a stator with permanent magnets and a slide with inductivities. The driving force is produced due to Lorentz's law. Another one is a hybrid stepper motor. Its stator is a comb-like structured ferromagnetic plate being magnetically passive. The permanent magnets and inductivities of the slide are arranged in such a way that the magnetic flux can be increased and decreased properly. The reluctance principle is used to produce the force in every motion direction.

In some special cases simplifications usually done in the theory of electrical machines can cause a mathematical problem. For instance, modelling of a 3-phase-machine with the same assumptions like in section 4 yields a singular matrix of inductivities. That's why the metric of the whole EMS is not regular and, if the simulation tool uses explicit integration codes only (like **alaska**), a new model of the 3-phase-machine has to be designed. This can bo done using a linear transformation with respect to generalized electrical coordinates.

The Lagrange approach allows to describe the "mechanical" and "electrical" subsystems of every discrete EMS and the electro-magneto-mechanical interactions between them in a unique way. Then, a coupling of different simulation tools can be avoided.

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